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Compact visibility representation of 4-connected plane graphs

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ABSTRACT

The *visibility representation* (VR for short) is a classical representation of plane graphs. The VR has various applications and has been extensively studied in the literature. A main focus of the study is to minimize the size of the VR. It is known that there exists a plane graph G with n vertices where any VR of G requires a size at least $\lfloor \frac{2n}{3} \rfloor \times (\lfloor \frac{4n}{3} \rfloor - 3)$. For upper bounds, it is known that every plane graph has a VR with size at most $\lfloor \frac{2}{3}n \rfloor \times (2n-5)$, and a VR with size at most $(n-1) \times \lfloor \frac{4}{3}n \rfloor$.

It has been an open problem to find a VR with both height and width simultaneously bounded away from the trivial upper bounds (namely of size $c_h n \times c_w n$ with $c_h < 1$ and $c_w < 2$). In this paper, we provide the first VR construction for a non-trivial graph class that simultaneously bounds both the height and the width. We prove that every 4-connected plane graph has a VR with height $\leq \frac{3n}{4} + 2\lceil \sqrt{n} \rceil + 4$ and width $\leq \lceil \frac{3n}{2} \rceil$.

Our VR algorithm is based on an *st*-orientation of 4-connected plane graphs with special properties. The area of the VR presented in this paper is larger than the area of some of the previous results for this graph class. However, bounding one dimension of the VR only requires finding a good *st*-orientation *or* a good dual s^*t^*-orientation of G. On the other hand, bounding both dimensions of the VR requires finding a good *st*-orientation *and* a good dual s^*t^*-orientation of G at the same time, and hence is far more challenging. Since the *st*-orientation is a very useful concept in other applications, this result may be of independent interest.

Reducing the height (the width, respectively) of the VR is the same as reducing the length of the longest path in an *st*-orientation of G (dual *st*-orientation, respectively). We show that it's NP-complete to find an *st*-orientation of a 2-connected plane graph that minimizes the sum of the length of the longest path in the orientation and the length of the longest path in its dual orientation.

1. Introduction

Drawing plane graphs has emerged as a fast growing research area in recent years (see [1] for a survey). A *visibility representation* (VR for short) of a plane graph G is a drawing of G, where the vertices of G are represented by non-overlapping horizontal line segments (with integer end point coordinates), and each edge of G is represented by a vertical line segment touching the segments of its end vertices. Fig. 1 shows a VR of a plane graph G. The problem of finding a compact VR is important not only in algorithmic graph theory, but also in practical applications. A simple linear time VR algorithm was given in [15,17] for 2-connected plane graphs. It uses an *st-orientation* of G and the corresponding *st*-orientation of its *st*-dual G^* to construct VR. Using this approach, the height of the VR is bounded by $(n-1)$ and the width of the VR is bounded by $(2n-5)$ [15,17].

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Table 1
Previous results on the height and the width of VR. For the line 8, the original bound given in [22] was Height $\leq 2n/3 + O(1)$. By a more careful calculation, the term $O(1)$ is actually 14. Line 10 is the results in this paper.

	Plane Graph		4-Connected Plane Graph	
	Width	Height	Width	Height
1	$\leq (2n-5)$ [15,17]	$\leq (n-1)$ [15,17]		
2	$\leq \lfloor \frac{3n-6}{2} \rfloor$ [7]			
3	$\leq \lfloor \frac{22n-42}{15} \rfloor$ [10]		$\leq (n-1)$ [8]	
4		$\leq \lfloor \frac{5n}{6} \rfloor$ [19]		
5	$\leq \lfloor \frac{13n-24}{9} \rfloor$ [20]			$\leq \lceil \frac{3n}{4} \rceil$ [18]
6		$\leq \lfloor \frac{4n-1}{5} \rfloor$ [21]		
7		$\leq \frac{2n}{3} + \lfloor 2\sqrt{n} \rfloor$ [6]		
8		$\leq \frac{2n}{3} + 14$ [22]		
9	$\leq \lfloor \frac{4n}{3} \rfloor - 2$ [4]			$\leq \lceil \frac{n}{2} \rceil + 2\lceil \sqrt{\frac{n-2}{2}} \rceil$ [3]
10			$\leq \frac{3}{2}n$	$\leq \frac{3}{4}n + 2\lceil \sqrt{n} \rceil + 4$

As in many other graph drawing problems, one of the main concerns in VR research is to minimize the size of the representation. For the lower bounds, it was shown in [19] that there exists a plane graph G with n vertices where any VR of G requires a size at least $\lfloor \frac{2n}{3} \rfloor \times (\lfloor \frac{4n}{3} \rfloor - 3)$. Several papers have been published to reduce the height and width of the VR by carefully constructing special st-orientations. Table 1 summarizes related previous results.

All these VR constructions concentrated on one dimension only. In the table above, the unmentioned dimension is bounded by the trivial upper bound ($n-1$ for the height and $2n-5$ for the width). In [12,13], heuristic algorithms were developed aiming at reducing the height and the width of VR simultaneously. It has been illusive to find a VR with both height and width simultaneously bounded away from the trivial upper bounds (namely of size $c_h n \times c_w n$ with $c_h < 1$ and $c_w < 2$).

In this paper, we prove that every 4-connected plane graph of n vertices has a VR with height $\leq \frac{3n}{4} + 2\lceil \sqrt{n} \rceil + 4$ and width $\leq \lceil \frac{3n}{2} \rceil$. The representation can be constructed in linear time. Many planar graphs can be triangulated into 4-connected planar graphs (including quadrangulations) [2]. Our VR can be used for these planar graphs also.

Reducing the height (the width, respectively) of VR is the same as reducing the length of the longest path in an st-orientation of G (dual st-orientation, respectively). The problem of finding an optimal st-orientation for general graphs has been shown to be NP-Hard by Gallai [5] and Papamanthou and Tollis [14]. The problem of finding an optimal st-orientation for plane graphs has been shown to be NP-Complete in [16]. In this paper, we show that it's NP-complete to find an st-orientation of a 2-connected plane graph that minimizes the sum of the length of the longest path in the orientation and the length of the longest path in its dual orientation.

The present paper is organized as follows. Section 2 introduces preliminaries. Section 3 presents the construction of the VR with the stated height and width bounds. In Section 4, we present our NP-completeness proof. Section 5 concludes the paper.

2. Preliminaries

In this section, we give definitions and preliminary results. Definitions not mentioned here are standard. In this paper, a graph means an undirected graph unless stated otherwise. For an undirected graph $G = (V, E)$, (u, v) denotes an edge of G. G is called a *directed graph* (digraph for short) if each edge of G is assigned a direction. For a directed graph, $u \to v$ denotes a directed edge from u to v.

A *planar graph* is a graph G such that the vertices can be drawn in the plane and the edges can be drawn as non-intersecting curves. Such a drawing is called a *plane embedding*. The drawing divides the plane into a number of connected regions. Each region is called a *face*. The unbounded face is the *exterior face*. Other faces are *interior faces*. A *plane graph* is a planar graph with a fixed embedding. A *plane triangulation* is a plane graph where every face is a triangle (including the exterior face). The vertices and the edges on the boundary of the exterior face are called *exterior vertices and edges*, respectively.

When discussing VR, we assume G is a plane triangulation. (If not, we get a plane triangulation G' by adding dummy edges into G. After constructing a VR for G', a VR of G is obtained by deleting the vertical line segments for the dummy edges.)

A *numbering* $\mathcal{O}$ of a set $S = \{a_1, \ldots, a_k\}$ is a one-to-one mapping between S and the set $\{1, 2, \ldots, k\}$. We write $\mathcal{O} = \langle a_{i_1}, a_{i_2}, \ldots, a_{i_k} \rangle$ to indicate $\mathcal{O}(a_{i_1}) = 1$, $\mathcal{O}(a_{i_2}) = 2 \ldots$ etc. A set S with a numbering written this way is called an *ordered list*. For two elements a_i and a_j, if a_i is assigned a smaller number than a_j in $\mathcal{O}$, we write $a_i \prec_{\mathcal{O}} a_j$. Let S_1 and S_2 be

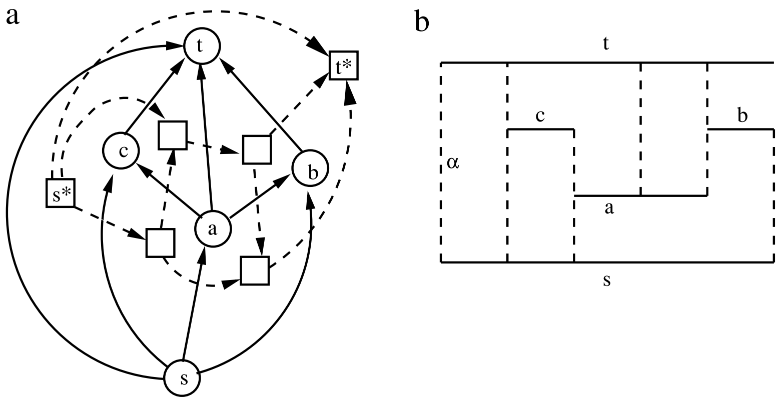

Fig. 1. (a) An st-graph G and its st-dual graph G^*; (b) A VR of G.

two disjoint sets. If $\mathcal{O}_1$ is a numbering of S_1 and $\mathcal{O}_2$ is a numbering of S_2, their concatenation $\mathcal{O} = \langle \mathcal{O}_1, \mathcal{O}_2 \rangle$ is the numbering of $S_1 \cup S_2$ where $\mathcal{O}(x) = \mathcal{O}_1(x)$ for all $x \in S_1$ and $\mathcal{O}(y) = \mathcal{O}_2(y) + |S_1|$ for all $y \in S_2$.

An *orientation* of an (undirected) graph G is a digraph obtained from G by assigning a direction to each edge of G. Let $G = (V, E)$ be an undirected graph. A numbering $\mathcal{O}$ of V induces an orientation of G as follows: each edge of G is directed from its lower numbered end vertex to its higher numbered end vertex. The resulting digraph, denoted by $G_{\mathcal{O}}$, is called the *orientation derived from* $\mathcal{O}$ which, obviously, is acyclic. We use length($G_{\mathcal{O}}$) (or simply length($\mathcal{O}$) if G is clear from the context) to denote the length of the longest path in $G_{\mathcal{O}}$. (The length of a path is the number of edges in it.)

Let G be a 2-connected plane graph with an exterior edge (s, t). An orientation of G is called an *st-orientation* if the resulting digraph is acyclic with s as the only source and t as the only sink. Such a digraph is also called an *st-graph*. Lempel et al. [9] showed that for every 2-connected plane graph G and an exterior edge (s, t), there exists an st-orientation. Properties of st-orientations can be found in [11].

Let G be a 2-connected plane graph and (s, t) an exterior edge. An *st-numbering* of G is a one-to-one mapping $\xi : V \to \{1, 2, \ldots, n\}$, such that $\xi(s) = 1$, $\xi(t) = n$, and each vertex $v \neq s, t$ has two neighbors u, w with $\xi(u) < \xi(v) < \xi(w)$. Given an st-numbering ξ of G, the orientation of G derived from ξ is obviously an st-orientation of G. On the other hand, if $G = (V, E)$ has an st-orientation $\mathcal{O}$, we can define a 1-1 mapping $\xi : V \to \{1, \ldots, n\}$ by topological sort of $G_{\mathcal{O}}$. It is easy to see that ξ is an st-numbering and the orientation derived from ξ is $\mathcal{O}$. From now on, we will interchangeably use the term “an st-numbering” of G and the term “an st-orientation” of G.

Definition 1. Let G be a plane graph with an st-orientation $\mathcal{O}$, where $s \to t$ is an exterior edge drawn at the left on the exterior face of G. The *st-dual* graph G^* of G and the dual orientation $\mathcal{O}^*$ of $\mathcal{O}$ is defined as follows:

- Each face f of G corresponds to a node f^* of G^*. The unique interior face adjacent to the edge (s, t) corresponds to a node s^* in G^*, the exterior face corresponds to a node t^* in G^*.
- For each edge $e \neq (s, t)$ of G separating a face f_1 on its left and a face f_2 on its right, there is a dual edge e^* in G^* from f_1^* to f_2^*.
- The dual edge of the exterior edge (s, t) is directed from s^* to t^*.

Fig. 1(a) shows an st-graph G and its st-dual graph G^* (where circles and solid lines denote the vertices and the edges of G; squares and dashed lines denote the nodes and the edges of G^*). It is well known that the st-dual graph G^* defined above is an st-graph with source s^* and sink t^*. The following theorem was proved in [15,17]:

Theorem 1. *Let G be a 2-connected plane graph with an st-orientation $\mathcal{O}$. Let $\mathcal{O}^*$ be the dual st-orientation of G^*. A VR of G can be obtained from $\mathcal{O}$ in linear time. The height of the VR is* length($\mathcal{O}$) $\leq n - 1$. *The width of the VR is* length($\mathcal{O}^*$) $\leq 2n - 5$ *(which is the number of nodes in G^*).*

Fig. 1(b) shows a VR of the graph G shown in Fig. 1(a). The width of the VR is length($\mathcal{O}^*$) $= 5$. The height of the VR is length($\mathcal{O}$) $= 3$.

Definition 2. A *4TP graph* is a plane graph G obtained from a 4 connected plane triangulation by deleting one of its exterior edges.

Note that every interior face of a 4PT graph G is a triangle and the exterior face of G is a quadrangle. The four exterior vertices of a 4TP graph G will be denoted by v_S, v_W, v_N, v_E in clockwise order. An st-orientation of G is an acyclic orientation of G so that v_S is the only source and v_N is the only sink. Let H be a 4-connected plane triangulation and $e = (s, t)$ an exterior edge of H. If we delete e from H, the resulting graph $G = H - \{e\}$ is a 4TP graph. We label the exterior vertices of G so that $s = v_S$ and $t = v_N$. Our algorithm will construct a VR $\mathcal{D}$ of G so that the line segment l_s for s has the smallest y-coordinate, and the line segment l_t for t has the largest y-coordinate. From $\mathcal{D}$, we can obtain a VR $\mathcal{D}'$ of H as follows: extend both l_s and

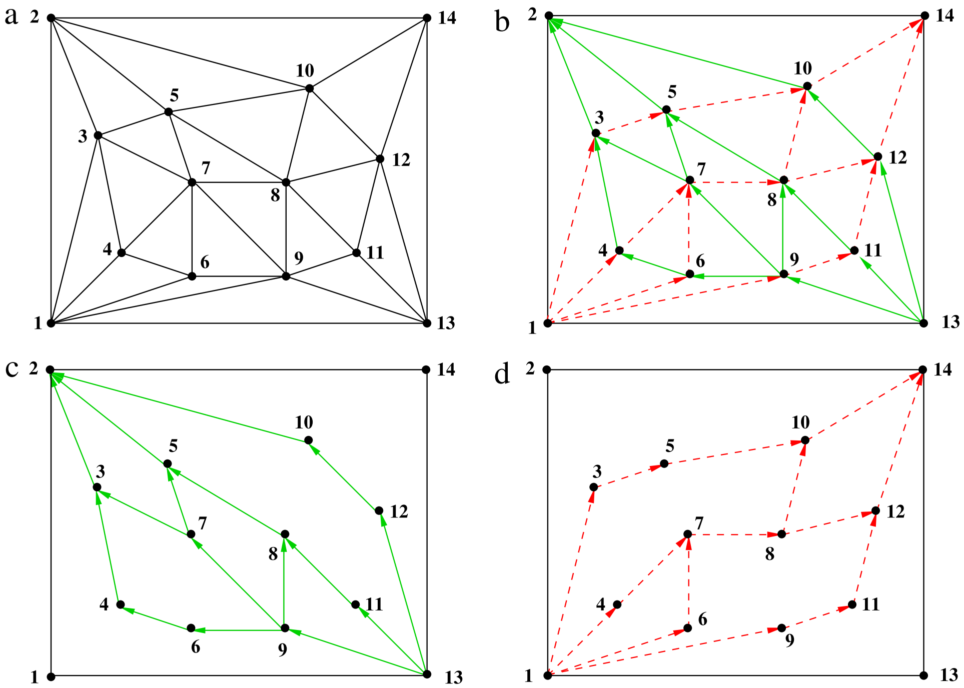

Fig. 2. (a) A 4TP graph G; (b) A REL $\mathcal{R}$ of G; (c) G_{green}; (d) G_{red}.

l_t to the left by one unit, then add a vertical line segment α connecting them (see Fig. 1). This operation does not change the height of $\mathcal{D}$ and increases the width of $\mathcal{D}$ by 1. From now on we will consider 4TP graphs only.

Definition 3. A *regular edge labeling* (REL) of a 4TP graph $G = (V, E)$ is a partition and orientation of the interior edges of G into two subsets E_{green}, E_{red} of directed edges such that:

(1) For each interior vertex v, the edges incident to v appear in clockwise order around v as follows: a set of edges in E_{red} leaving v; a set of edges in E_{green} entering v; a set of edges in E_{red} entering v; a set of edges in E_{green} leaving v.
(2) All interior edges incident to v_N are in E_{red} and entering v_N. All interior edges incident to v_W are in E_{green} and entering v_W. All interior edges incident to v_S are in E_{red} and leaving v_S. All interior edges incident to v_E are in E_{green} and leaving v_E.

Fig. 2(b) shows a REL of a 4TP graph G shown in Fig. 2(a). The green and red edges are drawn as solid and dashed lines respectively, where $v_S = 1$, $v_W = 2$, $v_N = 14$, $v_E = 13$. (The four exterior vertices will always be drawn at the lower-left, upper-left, upper-right and lower-right corners, respectively.) It was shown in [8] that every 4TP graph has a REL, constructable in linear time.

Let $\mathcal{R}$ be a REL of a 4TP graph G. Let $G_{\mathcal{R}}$ be the orientation of G obtained from $\mathcal{R}$ as follows. The interior edges are directed as in $\mathcal{R}$ (ignoring the colors). The exterior edges are oriented as: $v_S \to v_W$, $v_W \to v_N$, $v_S \to v_E$, $v_E \to v_N$. It was shown in [8] that $G_{\mathcal{R}}$ is an st-orientation of G with source v_S and sink v_N. $G_{\mathcal{R}}$ will be called the *st-orientation derived from* $\mathcal{R}$.

Let G_{green} (G_{red}, respectively) be the directed graph obtained from $G_{\mathcal{R}}$ by deleting all red (green, respectively) interior edges. Note that both G_{green} and G_{red} are st-graphs with source v_S and sink v_N. G_{green} and G_{red} are shown in Fig. 2(c) and (d). The internal faces of G_{green} and G_{red} are called *red faces* and the *blue faces*, respectively.

Lemma 1. *Let $f(G_{green})$ and $f(G_{red})$ be the number of* interior faces *of G_{green} and G_{red}, respectively. Then $f(G_{green})+f(G_{red}) = n-1$.*

Proof. By the Euler formula, a 4TP graph G with n vertices has $m = 3n - 7$ edges. Both G_{green} and G_{red} contain n vertices. Let m_r and m_g be the number of edges in G_{red} and in G_{green} respectively. By the Euler formula, we have: $f(G_{green}) = m_g + 1 - n$ and $f(G_{red}) = m_r + 1 - n$. Each interior edge of G belongs to either G_{green} or G_{red}. The four exterior edges belong to both G_{green} and G_{red}. Thus $m_g + m_r = m + 4$. Therefore $f(G_{green}) + f(G_{red}) = m_g + m_r + 2 - 2n = m + 4 + 2 - 2n = n - 1$. $\square$

In Fig. 2, G has $n = 14$ vertices, $f(G_{green}) = 7$ and $f(G_{red}) = 6$. The following lemma was proved in [8] by a complicated argument. A much simpler proof is given below. The argument used here will be useful later.

Lemma 2. *Let G be a 4TP graph with a REL $\mathcal{R}$. Let $G_{\mathcal{R}}$ be the st-orientation of G derived from $\mathcal{R}$. Let $G^*_{\mathcal{R}}$ be the corresponding dual st-orientation of G^*. Then* $\text{length}(G^*_{\mathcal{R}}) \le n - 1$. *In other words, the VR of G obtained from $G_{\mathcal{R}}$ has width $\le n - 1$.*

Proof. Let $P^* = \{e^*_{i_1}, e^*_{i_2}, \dots, e^*_{i_k}\}$ be a longest path in $G^*_{\mathcal{R}}$, where $k = \text{length}(G^*_{\mathcal{R}})$. For each j ($1 \le j \le k$), let e_{i_j} be the edge in G corresponding to $e^*_{i_j}$. If e_{i_j} is red, then when P^* passes $e^*_{i_j}$, it enters a new red face in G_{red}. If e_{i_j} is green, then when P^* passes $e^*_{i_j}$, it enters a new green face in G_{green}. Because both G_{red} and G_{green} are plane st-graphs, each red or green face can be entered at most once. By Lemma 1, P^* can visit at most $f(G_{green}) + f(G_{red}) + 1 = n$ faces. (The additional 1 is for the external face.) Therefore the length of P^* is at most $k \le n - 1$. $\square$

The following definition was used in [3,6] to find st-orientations with special properties.

Definition 4. A *ladder graph* of order n is a plane graph $L = (A \cup B, E_L)$. The vertex set of L can be partitioned into $A = \{a_1, \dots, a_{\lceil n/2 \rceil}\}$ and $B = \{b_1, \dots, b_{\lfloor n/2 \rfloor}\}$. $E_L = L_A \cup L_B \cup E_{cross}$ where:

- $L_A = \{(a_i, a_{i+1}) | 1 \le i < \lceil n/2 \rceil\}$; $L_B = \{(b_j, b_{j+1}) | 1 \le j < \lfloor n/2 \rfloor\}$.
- E_{cross} consists of edges, (called *cross edges* of L), between a vertex $a_i \in A$ and a vertex $b_j \in B$; no two edges in E_{cross} cross each other; and the edges (a_1, b_1), $(a_{\lceil n/2 \rceil}, b_{\lfloor n/2 \rfloor}) \in E_{cross}$.

For a cross edge (a_i, b_j), define $\text{slope}(a_i, b_j) = j - i$. It is called a *level* (or *up* or *down*, respectively) edge if $\text{slope}(a_i, b_j) = 0$ (or $\text{slope}(a_i, b_j) > 0$ or $\text{slope}(a_i, b_j) < 0$, respectively).

Definition 5. An orientation $\mathcal{L}$ of a ladder graph L is *consistent* if the following hold:

(1) For any i, the edge $a_i \to a_{i+1}$ is directed from a_i to a_{i+1}, and the edge $b_i \to b_{i+1}$ is directed from b_i to b_{i+1}.
(2) The edges in E_{cross} are oriented in a way such that $\mathcal{L}$ is acyclic.

From the definition, it is clear that a consistent orientation $\mathcal{L}$ is an st-orientation of L with source either a_1 or b_1, and sink either $a_{\lceil n/2 \rceil}$ or $b_{\lfloor n/2 \rfloor}$, depending on the orientations of the edges (a_1, b_1) and $(a_{\lceil n/2 \rceil}, b_{\lfloor n/2 \rfloor})$.

Theorem 2. *Every ladder graph L of order n has a consistent orientation $\mathcal{L}$, constructable in linear time, such that the following hold:*

(1) *a_1 is the only source and $b_{\lfloor n/2 \rfloor}$ is the only sink of $\mathcal{L}$.*
(2) $\text{length}(\mathcal{L}) \le \lceil n/2 \rceil + 2\lceil \sqrt{(n-2)/2} \rceil$.

The essentially same theorem was originally proved in [6]. The theorem stated above is adapted from a slightly different version in [3]. It can be proved by a slight modification of the proof in [3].

3. Compact VR of 4-connected plane graphs

In order to construct a VR of G with stated width and height, by Theorem 1, all we need is to find an st-orientation $\mathcal{O}$ of G so that both $\text{length}(\mathcal{O})$ and $\text{length}(\mathcal{O}^*)$ are not too large. The main difficulty of the construction is that these two goals often conflict. We will use a REL $\mathcal{R}$ of G to guide the construction of $\mathcal{O}$. (This is why we need the 4-connectivity: only 4PT graphs have REL.)

Throughout this section, $G = (V, E)$ denotes a 4TP graph and $\mathcal{R}$ a REL of G. The basic idea of the construction is as follows: first, we use $\mathcal{R}$ to partition G into two subgraphs G_A and G_B of equal size. In the st-orientation $\mathcal{O}$, the orientations of the edges within G_A and G_B are the same as in $\mathcal{R}$. The edges of G between G_A and G_B form a ladder graph E_{cross}. The crux for constructing $\mathcal{O}$ is to orient the edges in E_{cross} in order to bound both $\text{length}(\mathcal{O})$ and $\text{length}(\mathcal{O}^*)$.

3.1. Partition G into G_A, G_B and E_{cross}

Let $\mathcal{R}^{rev}$ denote the orientation of G obtained from $\mathcal{R}$ by reversing the direction of green edges in $\mathcal{R}$. Let $G_{\mathcal{R}^{rev}}$ be the orientation of G derived from $\mathcal{R}^{rev}$.

Lemma 3. *$G_{\mathcal{R}^{rev}}$ is an st-orientation of G.*

Proof. Observe that if we flip G through a line that passes v_S and v_N, then $\mathcal{R}^{rev}$ is just a REL of G (with the roles of v_E and v_W switched). $\square$

Let $\mathcal{P} = \langle v_1, v_2, \dots, v_n \rangle$ be a topological ordering of $G_{\mathcal{R}^{rev}}$. Then we have: $v_1 = v_S$, $v_2 = v_W$, $v_{n-1} = v_E$ and $v_n = v_N$. Partition V into two subsets: $A = \{v_1, v_2, \dots, v_{\lceil n/2 \rceil}\}$ and $B = \{v_{\lceil n/2 \rceil + 1}, \dots, v_n\}$. Let G_A (G_B, respectively) be the subgraph of G induced by the vertex set A (B, respectively). Let $G_{A\mathcal{R}}$ ($G_{B\mathcal{R}}$, respectively) denote the graph G_A (G_B, respectively) whose edges are partitioned and oriented according to $\mathcal{R}$.

Next, we order the vertex set of G_A as $A = \langle a_1, a_2, \dots, a_{\lceil n/2 \rceil} \rangle$ by a topological sort of $G_{A\mathcal{R}}$. Note that $a_1 = v_S$ and $a_{\lceil n/2 \rceil} = v_W$. Similarly, we order the vertex set of G_B as $B = \langle b_1, b_2, \dots, b_{\lfloor n/2 \rfloor} \rangle$ by a topological sort of $G_{B\mathcal{R}}$. Note that $b_1 = v_E$ and $b_{\lfloor n/2 \rfloor} = v_N$.

The edge set of G can be partitioned into three subsets: E_A is the edge set of G_A; E_B is the edge set of G_B; and E_{cross} is the set of edges between A and B. Let P_A be the path from a_1 to $a_{\lceil n/2 \rceil}$ on the exterior face of G_A. Let P_B be the path from b_1 to $b_{\lfloor n/2 \rfloor}$

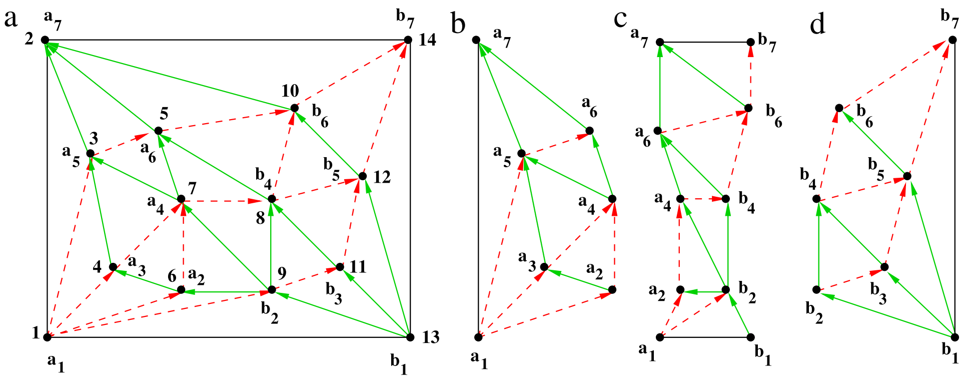

Fig. 3. (a) Numbering of G; (b) G_A; (c) E_{cross}; (d) G_B.

on the exterior face of G_B. Let C be the region bounded by P_A, P_B and the edges (a_1, b_1) and $(a_{\lceil n/2\rceil}, b_{\lfloor n/2\rfloor})$. The faces of G in the region C are called the *cross faces* of G.

Fig. 3(a) shows the numbering of the graph in Fig. 2(b). The vertices are numbered by a topological ordering of $G_{\mathcal{R}^{rev}}$. The labels $a_1, \ldots, a_7, b_1, \ldots, b_7$ indicate their numbering in $G_{A\mathcal{R}}$ and $G_{B\mathcal{R}}$ respectively. Fig. 3(b), (c) and (d) show the graph $G_{A\mathcal{R}}$, the edges in E_{cross} (with the paths P_A and P_B) and $G_{B\mathcal{R}}$ respectively. $P_A = \{a_1, a_2, a_4, a_6, a_7\}$ and $P_B = \{b_1, b_2, b_4, b_6, b_7\}$.

Lemma 4. (1) *The numbering* $\langle a_1, a_2, \ldots, a_{\lceil n/2\rceil}\rangle$ *is an st-numbering of* G_A.
(2) *The numbering* $\langle b_1, b_2, \ldots, b_{\lfloor n/2\rfloor}\rangle$ *is an st-numbering of* G_B.

Proof. We only prove (1). The proof of (2) is similar. Since $G_{\mathcal{R}}$ is acyclic and $G_{A\mathcal{R}}$ is a subgraph of $G_{\mathcal{R}}$, $G_{A\mathcal{R}}$ is acyclic. Clearly, $a_1 = v_S$ is a source and $a_{\lceil n/2\rceil} = v_W$ is a sink of $G_{A\mathcal{R}}$. Consider any vertex $v = a_i$ $(1 < i < \lceil n/2\rceil)$. We need to show a_i has two neighbors a_j and a_k with $j < i < k$.

Since v is an interior vertex of G, there is a red edge $e = u \to v$ in $\mathcal{R}$. e is oriented as $u \to v$ in $\mathcal{R}^{rev}$. Thus, in the topological ordering of $G_{\mathcal{R}^{rev}}$, u is numbered before v. (Namely $u = v_p$ and $v = v_q$ with $p < q$). Hence $u \in A$. Since e is directed as $u \to v$ in $G_{A\mathcal{R}}$, u is numbered before v in the topological ordering of $G_{A\mathcal{R}}$. Namely $u = a_j$ for some $j < i$.

Since v is an interior vertex of G, there is a green edge $e' = v \to w$ in $\mathcal{R}$. e' is oriented as $w \to v$ in $\mathcal{R}^{rev}$. Thus, in the topological ordering of $G_{\mathcal{R}^{rev}}$, w is numbered before v. (Namely $w = v_r$ and $v = v_q$ with $r < q$). Hence $w \in A$. Since e' is directed as $v \to w$ in $G_{A\mathcal{R}}$, w is numbered after v in the topological ordering of $G_{A\mathcal{R}}$. Namely $w = a_k$ for some $i < k$. □

Construct a ladder graph $L = (A \cup B, E_{cross})$ as follows: L contains a path $L_A = a_1 \to a_2 \ldots \to a_{\lceil n/2\rceil}$, a path $L_B = b_1 \to b_2 \ldots \to b_{\lfloor n/2\rfloor}$ and the edges in E_{cross}. (For the graph shown in Fig. 3(a), the corresponding ladder graph L can be obtained from the graph shown in Fig. 3(c) by inserting the vertex a_3 into the edge (a_2, a_4), a_5 into (a_4, a_6), b_3 into (b_2, b_4) and b_5 into (b_4, b_6).)

Definition 6. Let $\mathcal{L}$ be a consistent orientation of L. $G_{\mathcal{RL}}$ denotes the orientation of G obtained as follows: the edges in E_A and E_B are oriented as in $G_{\mathcal{R}}$. The edges in E_{cross} are oriented as in $\mathcal{L}$.

Lemma 5. *For any consistent orientation* $\mathcal{L}$ *of* L, $G_{\mathcal{RL}}$ *is an st-orientation of* G.

Proof. Since $G_{A\mathcal{R}}$ and $G_{B\mathcal{R}}$ are acyclic and $\mathcal{L}$ is a consistent orientation of L, $G_{\mathcal{RL}}$ is acyclic. Consider any interior vertex v of G. If $v \in A$, then v has two neighbors u, w in G_A such that $u \to v$ and $v \to w$ in $G_{A\mathcal{R}}$. If $v \in B$, then v has two neighbors u, w in G_B such that $u \to v$ and $v \to w$ in $G_{B\mathcal{R}}$. Depending on the orientation of the cross edge (a_1, b_1) in $\mathcal{L}$, either a_1 or b_1 is the unique source of $G_{\mathcal{RL}}$. Depending on the orientation of the cross edge $(a_{\lceil n/2\rceil}, b_{\lfloor n/2\rfloor})$ in $\mathcal{L}$, either $a_{\lceil n/2\rceil}$ or $b_{\lfloor n/2\rfloor}$ is the unique sink of $G_{\mathcal{RL}}$. Thus $G_{\mathcal{RL}}$ is an st-orientation of G. □

Let $\mathcal{L}$ be any consistent orientation of L. $L_{\mathcal{L}}$ denotes the st-orientation of L derived from $\mathcal{L}$. Let $G_{\mathcal{RL}}$ be the st-orientation of G derived from $\mathcal{R}$ and $\mathcal{L}$, and $G^*_{\mathcal{RL}}$ the corresponding dual st-orientation of G^*.

Theorem 3. $\text{length}(G_{\mathcal{RL}}) \le \text{length}(L_{\mathcal{L}})$.

Proof. Let P be a longest path in $G_{\mathcal{RL}}$. We transform P to a path P_L in $L_{\mathcal{L}}$ as follows. Consider any edge $e = u \to v$ in P. If e is a cross edge, we keep it in P_L. If e is an edge in G_A, then $u = a_i$ and $v = a_j$ for some $i < j$. We replace e by the sub-path in L_A from a_i to a_j. If e is in G_B, we replace it by a sub-path in L_B. After this operation is performed to all edges in P, we obtain a directed path P_L in $L_{\mathcal{L}}$. Thus: $\text{length}(G_{\mathcal{RL}}) = \text{length}(P) \le \text{length}(P_L) \le \text{length}(L_{\mathcal{L}})$. □

Theorem 4. *Let* P^* *be a longest path in* $G^*_{\mathcal{RL}}$. *Then* $\text{length}(P^*) \le n - 1 + l$, *where* l *is the number of cross faces of* G *visited by* P^*.

Proof. Because the way $G_{\mathcal{RL}}$ is oriented, P^* travels in one of the following ways:

I: (i) P^* first crosses the edge $a_1 \to a_{\lceil n/2 \rceil}$ then visits some faces within G_A; (ii) crosses an edge in P_A and travels some l cross faces; (iii) crosses an edge in P_B, and travels some faces within G_B.

II: P^* first crosses the edge $a_{\lceil n/2 \rceil} \to b_{\lfloor n/2 \rfloor}$. It is similar to Case I, except the portion (i) is empty.

We count the length of these sub-paths separately.

G_A has $n_a = \lceil n/2 \rceil$ vertices. Let o_a be the number of vertices on the exterior face of G_A. Let $i_a = n_a - o_a$ be the number of interior vertices of G_A. It is easy to show that the number of interior faces of G_A is $f_a = 2n_a - o_a - 2$. By the Euler formula, the number of edges in G_A is $m_a = n_a + f_a - 1 = n_a + (2n_a - o_a - 2) - 1 = 3n_a - o_a - 3$.

Let $G_{A,red}$ be the subgraph of G_A consisting of the exterior edges of G_A and its red interior edges. $G_{A,red}$ has n_a vertices. Let $m_{A,red}$ be the number of edges in $G_{A,red}$. By the Euler formula, the number on interior faces in $G_{A,red}$ is $f_{A,red} = m_{A,red} - n_a + 1$.

Let $G_{A,green}$ be the subgraph of G_A consisting of the exterior edges of G_A and its green interior edges. $G_{A,green}$ has n_a vertices, Let $m_{A,green}$ be the number of edges in $G_{A,green}$. By the Euler formula, the number on interior faces in $G_{A,green}$ is $f_{A,green} = m_{A,green} - n_a + 1$.

Since each of the o_a exterior edges of G_A belongs to both $G_{A,red}$ and $G_{A,green}$, we have $m_{A,red} + m_{A,green} = m_a + o_a$. Thus:

$f_{A,red} + f_{A,green} = (m_{A,red} - n_a + 1) + (m_{A,green} - n_a + 1) = m_{A,red} + m_{A,green} - 2n_a + 2 = m_a + o_a - 2n_a + 2 = (3n_a - o_a - 3) + o_a - 2n_a + 2 = n_a - 1 = \lceil n/2 \rceil - 1$.

(For example, the graph G_A in Fig. 3(b) has $n_a = 7$ vertices. $i_a = 2$, $o_a = 5$, $m_a = 3n_a - o_a - 3 = 13$, and $f_a = 2n_a - o_a - 2 = 7$. $f_{A,red} = 3$ and $f_{A,green} = 3$.)

Consider the sub-path of P^* when it travels within G_A. When P^* crosses a red edge, it enters a new face in $G_{A,red}$. When P^* crosses a green edge, it enters a new face in $G_{A,green}$. Since the edges in $G_{A\mathcal{R}}$ are oriented according to REL $\mathcal{R}$, each face in $G_{A,red}$ and $G_{A,green}$ can be entered at most once. Therefore the length of P^* within G_A is at most $f_{A,red} + f_{A,green} \leq \lceil n/2 \rceil - 1$.

Then P^* crosses an edge in P_A, and enters the first cross face. It continues to travel l cross faces. Then P^* crosses an edge in P_B (which adds 1 to the length of P^*) and enters the first face in G_B. By the same argument for the sub-path of P^* within G_A, the length of P^* within G_B is at most $n_b - 1 = \lfloor n/2 \rfloor - 1$. Hence length$(P^*) \leq (\lceil n/2 \rceil - 1) + l + 1 + (\lfloor n/2 \rfloor - 1) = n - 1 + l$. □

3.2. Orientation of E_{cross}

We next describe how to find a consistent orientation $\mathcal{L}$ for L so that the length of a longest path P in the st-orientation $G_{\mathcal{RL}}$ and the length of a longest path P^* in the dual st-orientation $G^*_{\mathcal{RL}}$ are not too large.

Let $k = |E_{cross}|$. Then G has $k - 1$ cross faces. Note that we always have $k \leq (n - 1)$. (The equality holds when all vertices of G_A are in the exterior path P_A and all vertices of G_B are in the exterior path P_B. In this case, P_A has $\lceil n/2 \rceil - 1$ edges and P_B has $\lfloor n/2 \rfloor - 1$ edges. Since each cross face consumes one edge in either P_A or P_B, there are $\lceil n/2 \rceil + \lfloor n/2 \rfloor - 2 = n - 2$ cross faces.)

Order the edges in E_{cross} from bottom up: $E_{cross} = \{e_1, e_2, \ldots, e_k\}$. Suppose that $e_t = (a_{i_t}, b_{j_t})$ for $1 \leq t \leq k$. In particular $e_1 = (a_1, b_1)$ and $e_k = (a_{\lceil n/2 \rceil}, b_{\lfloor n/2 \rfloor})$.

Consider any cross edge e_t. If e_t is an up edge (namely slope$(e_t) > 0$), it's natural to orient e_i as $a_{i_t} \to b_{j_t}$, because otherwise the length of P might increase. However, if we always orient cross edges this way, the length of P^* might be too large. (Consider a special case where all cross edges are up edges. If we orient all cross edges from A side to B side, then P^* may visit all cross faces). To avoid this, some up edge e_t might have to be oriented as $a_{i_t} \leftarrow b_{j_t}$. This, of course, might increase the length of P. The trick is to orient the cross edges in a way so that the lengths of P and P^* do not increase too much.

Let $p = \lfloor k/2 \rfloor$. Consider the edge $e_p = (a_{i_p}, b_{j_p})$. Without loss of generality, we assume slope$(e_p) = j_p - i_p \geq 0$. (If not, switch the roles of G_A and G_B.) There are several cases.

Case 1: slope$(e_p) = j_p - i_p \leq n/4$.

Case 1a: $e_{p+1} = (a_{i_{p+1}}, b_{j_{p+1}})$ with $j_{p+1} \neq j_p$ and $i_{p+1} = i_p$ (see Fig. 4(a)).

We divide L into three sub-ladder graphs:

- $X = (A_X \cup B_X, E_X)$ of order $x = 2(i_p - 1)$, $A_X = \{a_1, \ldots, a_{i_p - 1}\}$ and $B_X = \{b_1, \ldots, b_{i_p - 1}\}$.
  Let $\mathcal{L}_X$ be the consistent orientation of X in Theorem 2, with length$(\mathcal{L}_X) \leq \lceil x/2 \rceil + 2\lceil \sqrt{x - 2)/2} \rceil$.
- $Y = (A_Y \cup B_Y, E_Y)$ of order $y = 2(j_p - i_p + 1)$, $A_Y = \{a_{i_p}, \ldots a_{j_p}\}$ and $B_Y = \{b_{i_p}, \ldots, b_{j_p}\}$.
  Define $\mathcal{L}_Y = \langle b_{i_p}, b_{i_p+1}, \ldots, b_{j_p}, a_{i_p}, a_{i_p+1}, \ldots, a_{j_p} \rangle$. Note that length$(\mathcal{L}_Y) = y - 1$ and $y/2 = j_p - i_p + 1 =$ slope$(e_p) + 1 \leq n/4 + 1$.
- $Z = (A_Z \cup B_Z, E_Z)$ of order $z = n - 2j_p$, $A_Z = \{a_{j_p+1}, \ldots, a_{\lceil n/2 \rceil}\}$ and $B_Z = \{b_{j_p+1}, \ldots, b_{\lfloor n/2 \rfloor}\}$.
  Let $\mathcal{L}_Z$ be the consistent orientation of Z in Theorem 2, with length$(\mathcal{L}_Z) \leq \lceil z/2 \rceil + 2\lceil \sqrt{(z - 2)/2} \rceil$.

Define $\mathcal{L} = \langle \mathcal{L}_X, \mathcal{L}_Y, \mathcal{L}_Z \rangle$.

Case 1b: $e_{p+1} = (a_{i_{p+1}}, b_{j_{p+1}})$ with $j_{p+1} = j_p$ and $i_{p+1} \neq i_p$ (see Fig. 4(b)).

We divide L into three sub-ladder graphs:

- $X = (A_X \cup B_X, E_X)$ of order $x = 2i_p$, $A_X = \{a_1, a_2, \ldots, a_{i_p}\}$ and $B_X = \{b_1, b_2, \ldots, b_{i_p}\}$.
  Let $\mathcal{L}_X$ be the consistent ordering for X in Theorem 2, with length$(\mathcal{L}_X) \leq \lceil x/2 \rceil + 2\lceil \sqrt{(x - 2)/2} \rceil$.

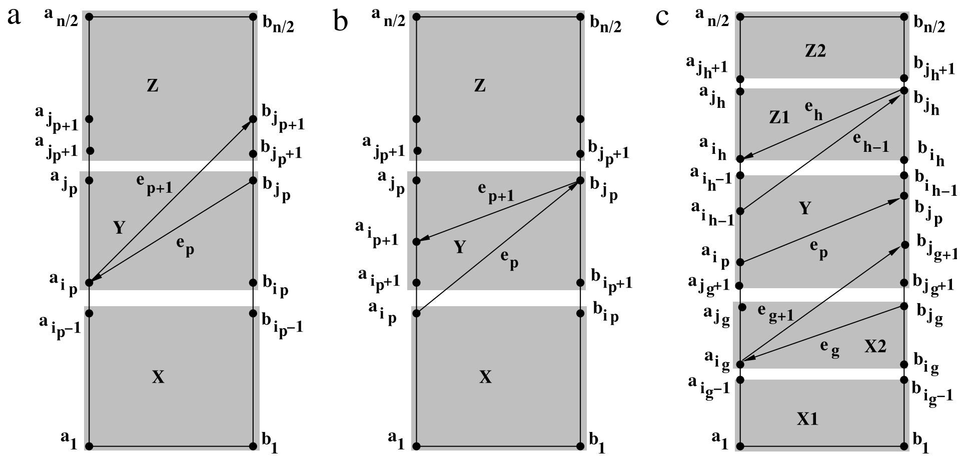

Fig. 4. (a) Case 1a; (b) Case 1b; (c) Case 2.

- $Y = (A_Y \cup B_Y, E_Y)$ of order $y = 2(j_p - i_p)$, $A_Y = \{a_{i_p+1}, \dots a_{j_p}\}$ and $B_Y = \{b_{i_p+1}, \dots, b_{j_p}\}$.
  Define $\mathcal{L}_Y = \langle b_{i_p+1}, \dots, b_{j_p}, a_{i_p+1}, \dots, a_{j_p}\rangle$.
- $Z = (A_Z \cup B_Z, E_Z)$ of order $z = n - 2j_p$, $A_Z = \{a_{j_p+1}, \dots, a_{\lceil n/2\rceil}\}$ and $B_Z = \{b_{j_p+1}, \dots, b_{\lfloor n/2\rfloor}\}$.
  Let $\mathcal{L}_Z$ be the consistent ordering for Z in Theorem 2, with length$(\mathcal{L}_Z) \le \lceil z/2\rceil + 2\lceil\sqrt{(z-2)/2}\rceil$.

Define $\mathcal{L} = \langle \mathcal{L}_X, \mathcal{L}_Y, \mathcal{L}_Z\rangle$.

Case 2: slope$(e_p) = j_p - i_p > n/4$ (see Fig. 4(c)).

Let $e_g = (a_{i_g}, b_{j_g})$ be the edge such that g is the largest index between 1 and $p-1$ with slope$(e_g) = j_g - i_g \le \frac{n}{4}$. (This edge exists because slope$(e_1) = 1 - 1 = 0$.) Let $e_h = (a_{i_h}, b_{j_h})$ be the edge such that h is the smallest index between $p+1$ and k with slope$(e_h) = j_h - i_h \le \frac{n}{4}$. (This edge exists because slope$(e_k) = \lfloor n/2\rfloor - \lceil n/2\rceil \le 0$.) We divide L into five sub-ladder graphs:

- $X1 = (A_{X1} \cup B_{X_1}, E_{X1})$ of order $x_1 = 2(i_g - 1)$, $A_{X1} = \{a_1, \dots, a_{i_g-1}\}$ and $B_{X1} = \{b_1, \dots, b_{i_g-1}\}$. Let $\mathcal{L}_{X1}$ be the consistent orientation for $X1$ in Theorem 2, with length$(\mathcal{L}_{X1}) \le \lceil x_1/2\rceil + 2\lceil\sqrt{(x_1-2)/2}\rceil$.
- $X2 = (A_{X2} \cup B_{X2}, E_{X2})$ of order $x_2 = 2(j_g - i_g + 1)$, $A_{X1} = \{a_{i_g}, a_{i_g+1}, \dots, a_{j_g}\}$ and $B_{X2} = \{b_{i_g}, b_{i_g+1}, \dots, b_{j_g}\}$. Define $\mathcal{L}_{X2} = \langle b_{i_g}, b_{i_g+1}, \dots, b_{j_g}, a_{i_g}, a_{i_g+1}, \dots, a_{j_g}\rangle$.
- $Y = (A_Y \cup B_Y, E_Y)$ of order $y = 2(i_h - j_g - 1)$, $A_Y = \{a_{j_g+1}, a_{j_g+2}, \dots, a_{i_h-1}\}$ and $B_Y = \{b_{j_g+1}, b_{j_g+2}, \dots, b_{i_h-1}\}$. Define $\mathcal{L}_Y = \langle a_{j_g+1}, a_{j_g+2}, \dots, a_{i_h-1}, b_{j_g+1}, b_{j_g+2}, \dots, b_{i_h-1}\rangle$.
- $Z1 = (A_{Z_1} \cup B_{Z1}, E_{Z1})$ of order $z_1 = 2(j_h - i_h + 1)$, $A_{Z1} = \{a_{i_h}, \dots, a_{j_h}\}$ and $B_{Z1} = \{b_{i_h}, \dots, b_{j_h}\}$. Define $\mathcal{L}_{Z1} = \langle b_{i_h}, b_{i_h+1}, \dots, b_{j_h}, a_{i_h}, a_{i_h+1}, \dots, a_{j_h}\rangle$.
- $Z2 = (A_{Z2} \cup B_{Z2}, E_{Z2})$ of order $z_2 = n - 2j_h$, $A_{Z2} = \{a_{j_h+1}, \dots, a_{\lceil n/2\rceil}\}$ and $B_{Z2} = \{b_{j_h+1}, \dots, b_{\lfloor n/2\rfloor}\}$. Let $\mathcal{L}_{Z2}$ be the consistent orientation for $Z2$ in Theorem 2, with length$(\mathcal{L}_{Z2}) \le \lceil z_2/2\rceil + 2\lceil\sqrt{(z_2-2)/2}\rceil$.

Define $\mathcal{L} = \langle \mathcal{L}_{X1}, \mathcal{L}_{X2}, \mathcal{L}_Y, \mathcal{L}_{Z1}, \mathcal{L}_{Z2}\rangle$.

We need the following lemmas for the analysis of Case 2.

Lemma 6. *There are at most $\lfloor n/2\rfloor$ cross faces between e_g and e_h.*

Proof. Consider the cross edge e_{g+1}. By the choice of e_g, we have slope$(e_{g+1}) > n/4$. So $e_{g+1} = (a_{i_g}, b_{j_{g+1}})$ for some $j_{g+1} > j_g$ and $j_{g+1} - i_g > n/4$. This implies $j_{g+1} > n/4 + i_g \ge n/4 + 1$.

Consider the cross edge e_{h-1}. By the choice of e_h, we have slope$(e_{h-1}) > n/4$. Thus $e_{h-1} = (a_{i_{h-1}}, b_{j_h})$ for some $i_{h-1} < i_h$ and $j_h - i_{h-1} > n/4$. This implies $i_{h-1} < j_h - n/4 \le \lfloor n/2\rfloor - n/4$.

Each cross face between e_{g+1} and e_{h-1} consumes either one edge in P_A between a_{i_g} and $a_{i_{h-1}}$, or one edge in P_B between $b_{j_{g+1}}$ and b_{j_h}. Hence the number of these cross faces is at most:

$$w \le (j_h - j_{g+1}) + (i_{h-1} - i_g) < (\lfloor n/2\rfloor - n/4 - 1) + (\lfloor n/2\rfloor - n/4 - 1) = 2\lfloor n/2\rfloor - n/2 - 2 \le \lfloor n/2\rfloor - 2.$$

Thus the number of cross faces between e_g and e_h is at most $w + 2 \le \lfloor n/2\rfloor$. □

Lemma 7. *Let $U = X2 \cup Y \cup Z1$. Let $\mathcal{L}_U = \langle \mathcal{L}_{X_2}, \mathcal{L}_Y, \mathcal{L}_{Z1}\rangle$. Then*

$$\text{length}(\mathcal{L}_U) \le (x_2 + y + z_1)/2 + n/4 + 1.$$

Proof. Let $S = \{e_{g+1}, e_{g+2}, \ldots, e_{h-2}, e_{h-1}\}$. Note that for any $e_t \in S$, $\text{slope}(e_t) > n/4$. Let P be a longest path in $\mathcal{L}_U$. Since b_{i_g} is the source and a_{j_h} is the sink in $\mathcal{L}_U$, P must start at b_{i_g} and end at a_{j_h}. The following are three different ways that P may achieve the maximum length. (In the following, the symbol $\stackrel{l}{\Longrightarrow}$ means a sub-path of length l. The symbol $\rightarrow$ means a single edge.)

- Q_1: $b_{i_g} \stackrel{x_2/2-1}{\Longrightarrow} b_{j_g} \rightarrow a_{i_g} \stackrel{j_h-i_g}{\Longrightarrow} a_{j_h}$.
  Then $\text{length}(Q_1) = (x_2/2-1)+1+(j_h-i_g) \le (x_2+y+z_1)/2+n/4$. (Here we use the facts: $j_h-i_g = (x_2+y+z_1)/2-1$ and $x_2/2 = j_g - i_g + 1 = \text{slope}(e_g) + 1 \le n/4 + 1$).
- Q_2: $b_{i_g} \stackrel{j_h-i_g}{\Longrightarrow} b_{j_h} \rightarrow a_{i_h} \stackrel{z_1/2-1}{\Longrightarrow} a_{j_h}$.
  By the same argument, we can show $\text{length}(Q_2) \le (x_2 + y + z_1)/2 + n/4$.
- $Q_3 = b_{i_g} \stackrel{x_2/2-1}{\Longrightarrow} b_{j_g} \rightarrow a_{i_g} \stackrel{i_t-i_g}{\Longrightarrow} a_{i_t} \stackrel{e_t}{\rightarrow} b_{j_t} \stackrel{j_h-j_t}{\Longrightarrow} b_{j_h} \rightarrow a_{i_h} \stackrel{j_h-i_h}{\Longrightarrow} a_{j_h}$
  Then: $\text{length}(Q_3) = (x_2/2-1)+1+(i_t-i_g)+1+(j_h-j_t)+1+(j_h-i_h) = (x_2/2+2)+(j_h-i_g)+[(j_h-i_h)-(j_t-i_t)]$.
  Note that $x_2/2 \le n/4 + 1$, $\text{slope}(e_h) = j_h - i_h \le n/4$ and $\text{slope}(e_t) = j_t - i_t > n/4$. Because $\text{slope}(e_h)$ and $\text{slope}(e_t)$ are integers, $[(j_h - i_h) - (j_t - i_t)] \le -1$. Thus:
  $\text{length}(Q_3) \le n/4 + 3 + (x_2 + y + z_1)/2 - 1 - 1 = (x_2 + y + z_1)/2 + n/4 + 1$. □

3.3. Analysis

Let $\mathcal{L}$ be the orientation for the ladder graph L constructed above.

Lemma 8. *$\mathcal{L}$ is a consistent orientation of L.*

Proof. We prove the lemma for Case 2. The proof for Cases 1a and 1b are similar. All we have to do is to show $\mathcal{L}$ is acyclic. The sub-orientations $\mathcal{L}_{X1}$, $\mathcal{L}_{Z2}$ are acyclic by Theorem 2. The sub-orientations $\mathcal{L}_{X2}$, $\mathcal{L}_Y$, $\mathcal{L}_{Z1}$ are acyclic by the construction. Since $\mathcal{L}$ is the concatenation of $\mathcal{L}_{X1}$, $\mathcal{L}_{X2}$, $\mathcal{L}_Y$, $\mathcal{L}_{Z1}$, $\mathcal{L}_{Z2}$, the orientations of the edges whose end vertices belong to different sub-ladder graphs do not create cycles. Hence $\mathcal{L}$ is acyclic. □

Note that in all cases, a_1 is the source and $b_{\lfloor n/2 \rfloor}$ is the sink of $\mathcal{L}$. Let $G_{\mathcal{RL}}$ be the st-orientation of G derived from $\mathcal{R}$ and $\mathcal{L}$. Let $G^*_{\mathcal{RL}}$ be the corresponding dual st-orientation of G^*.

Lemma 9. $\text{length}(G^*_{\mathcal{RL}}) \le \lceil \frac{3n}{2} \rceil - 1$.

Proof. Let P^* be a longest path in $G^*_{\mathcal{RL}}$. Let l be the number of cross faces visited by P^*. By Theorem 4, it is enough to show $l \le \lceil n/2 \rceil$.

Case 1a: Because the cross edges e_p and e_{p+1} are oriented in the opposite direction, P^* can visit the cross faces either in the region above e_p or in the region below e_{p+1}, but not both. Because each of these two regions has at most $\lceil (k-1)/2 \rceil + 1 \le \lceil (n-2)/2 \rceil + 1$ cross faces, we have $l \le \lceil n/2 \rceil$.

Case 1b: Similar to Case 1a.

Case 2: Note that e_g is oriented as $a_{i_g} \leftarrow b_{j_g}$ (see Fig. 4(c)). By the choice of e_g, $\text{slope}(e_{g+1}) > n/4$. Hence $e_{g+1} = (a_{i_g}, b_{j_{g+1}})$ for some $j_{g+1} > j_g$. So e_{g+1} is oriented $a_{i_g} \rightarrow b_{j_{g+1}}$ in $\mathcal{L}$. Similarly, we can show e_h is oriented as $a_{i_h} \leftarrow b_{j_h}$, and e_{h-1} is oriented as $a_{i_{h-1}} \rightarrow b_{j_h}$ for some $i_{h-1} < i_h$. Because of the orientations of the cross edges e_g, e_{g+1}, e_{h-1} and e_h, the path P^* can visit cross faces in only one of the following three regions:

- The region below the edge e_{g+1}. The number of cross faces in this region is at most $\lceil n/2 \rceil$ because this region is below e_p.
- The region between e_g and e_h. The number of cross faces in this region is at most $\lceil n/2 \rceil$ by Lemma 6.
- The region above e_{h-1}. The number of cross faces in this region is at most $\lceil n/2 \rceil$ because this region is above e_p. □

Lemma 10. $\text{length}(G_{\mathcal{RL}}) \le \frac{3n}{4} + 2\lceil \sqrt{n} \rceil + 4$.

Proof. Let P be a longest path in $L_{\mathcal{L}}$. By Theorem 3, it's enough to show $\text{length}(P) \le \frac{3n}{4} + 2\lceil \sqrt{n} \rceil + 4$.

Case 1: Let P_X, P_Y and P_Z be the sub-paths of P in the sub-ladder graphs X, Y and Z, respectively.

By Theorem 2, $\text{length}(P_X) \le \lceil x/2 \rceil + 2\lceil \sqrt{(x-2)/2} \rceil$ and $\text{length}(P_Z) \le \lceil z/2 \rceil + 2\lceil \sqrt{(z-2)/2} \rceil$. Since Y contains y vertices, $\text{length}(P_Y) \le y - 1$. The edges connecting these three sub-paths add 2 to the length of P. Noting the facts: $x + y + z = n$ and $y/2 \le n/4 + 1$, we have:

$$\begin{aligned}
\text{length}(P) &= \text{length}(P_X) + \text{length}(P_Y) + \text{length}(P_Z) + 2 \\
&\le (\lceil x/2 \rceil + 2\lceil \sqrt{(x-2)/2} \rceil) + (y-1) + (\lceil z/2 \rceil + 2\lceil \sqrt{(z-2)/2} \rceil) + 2 \\
&\le (x/2 + 1/2 + 2\lceil \sqrt{(x-2)/2} \rceil) + (y-1) + (z/2 + 1/2 + 2\lceil \sqrt{(z-2)/2} \rceil) + 2 \\
&= n/2 + y/2 + 2\lceil \sqrt{(x-2)/2} \rceil + 2\lceil \sqrt{(z-2)/2} \rceil + 2.
\end{aligned}$$

Let $f(x, z) = 2\lceil\sqrt{(x-2)/2}\rceil + 2\lceil\sqrt{(z-2)/2}\rceil$. Since $x+z \le n$, it is easy to check $f(x, z)$ reaches the maximum value when $x = z = n/2$: $f(n/2, n/2) \le 2\lceil\sqrt{n}\rceil$. Hence: $\text{length}(P) \le 3n/4 + 2\lceil\sqrt{n}\rceil + 3$.

Case 1b: Similar to Case 1a.

Case 2: Let $U = X2 \cup Y \cup Z1$ and $\mathcal{L}_U = \langle \mathcal{L}_{X_2}, \mathcal{L}_Y, \mathcal{L}_{Z1}\rangle$ (as in Lemma 7).

Let P_{X1}, P_U, P_{Z2} be the sub-paths of P in the sub-ladder graphs $X1$, U, and $Z2$ respectively.

By Theorem 2, $\text{length}(P_{X1}) \le \lceil x_1/2\rceil + 2\lceil\sqrt{(x_1-2)/2}\rceil$ and $\text{length}(P_{Z2}) \le \lceil z_2/2\rceil + 2\lceil\sqrt{(z_2-2)/2}\rceil$. By Lemma 7, $\text{length}(P_U) \le \text{length}(\mathcal{L}_U) \le (x_2 + y + z_1)/2 + n/4 + 1$. The edges connecting these 3 sub-paths add 2 to the length of P. Noting the facts that: $x_1 + x_2 + y + z_1 + z_2 = n$, $x_1 + z_2 \le n$, we have:

$$\begin{aligned}\text{length}(P) &= \text{length}(P_{X1}) + \text{length}(P_U) + \text{length}(P_{Z2}) + 2\\ &\le (\lceil x_1/2\rceil + 2\lceil\sqrt{(x_1-2)/2}\rceil) + (x_2 + y + z_1)/2 + n/4 + 1 + (\lceil z_2/2\rceil + 2\lceil\sqrt{(z_2-2)/2}\rceil) + 2\\ &\le n/2 + n/4 + 2\lceil\sqrt{(x_1-2)/2}\rceil + 2\lceil\sqrt{(z_2-2)/2}\rceil + 4 \le 3n/4 + 2\lceil\sqrt{n}\rceil + 4. \quad \square\end{aligned}$$

Now we can prove our main theorem.

Theorem 5. *Let H be a 4-connected plane graph with n vertices. Then H has a VR $\mathcal{D}$ with height $\le \frac{3n}{4} + 2\lceil\sqrt{n}\rceil + 4$ and width $\le \lceil 3n/2\rceil$. $\mathcal{D}$ can be constructed in linear time.*

Proof. Without loss of generality we assume G is a triangulation. We have the following algorithm.

(1) Delete an exterior edge $e = (v_S, v_N)$ from H. The resulting graph G is a 4TP graph.
(2) Find a REL $\mathcal{R}$ of G in $O(n)$ time [8].
(3) Partition G into G_A and G_B by a topological sort of $G_{\mathcal{R}^{rev}}$.
(4) Construct the ladder graph L and find the consistent orientation $\mathcal{L}$ for L as in Section 3.2.
(5) Construct the st-orientation $G_{\mathcal{R}\mathcal{L}}$. Find a VR $\mathcal{D}'$ for G in linear time by Theorem 1. (The construction ensures that the line segment l_S for v_S has the smallest y-coordinate, and the line segment l_N for v_N has the largest y-coordinate.)
(6) Add a vertical line for the deleted edge, we get a VR $\mathcal{D}$ of H.

By Lemmas 9 and 10, the VR $\mathcal{D}'$ for G has height $\le \frac{3n}{4} + 2\lceil\sqrt{n}\rceil + 4$ and width $\le \lceil 3n/2\rceil - 1$. The last step increases the width of $\mathcal{D}$ by 1. So the height and the width of $\mathcal{D}$ satisfy the stated bounds. All steps of the algorithm can be done in linear time. So the total run time is $O(n)$. $\square$

4. NP completeness proof

The following problem was proved to be NPC in [16].

Optimal st-orientation problem

Instance: a 2-connected plane graph G with an exterior edge (s, t), and a positive integer K.

Question: does G have an st-orientation $\mathcal{O}$ such that $\text{length}(\mathcal{O}) \le K$?

Because of the one-to-one correspondence between the st-orientations $\mathcal{O}$ of G and its dual st-orientation $\mathcal{O}^*$ of G^*, one can easily check that the following problem is also NP-Complete:

Optimal length dual st-orientation problem

Instance: a 2-connected plane graph G with an exterior edge (s, t), and a positive integer K.

Qustion: does G^* have an st-orientation $\mathcal{O}^*$ (with source s^* and sink t^*) such that $\text{length}(\mathcal{O}^*) \le K$?

We abbreviate this problem as *Dual-ST-Length*. Next consider the following decision problem:

Optimal sum st-orientation problem

Instance: a 2-connected plane graph G with an exterior edge (s, t), and a positive integer K.

Question: does G have an st-orientation $\mathcal{O}$ such that $\text{length}(\mathcal{O}) + \text{length}(\mathcal{O}^*) \le K$?

We abbreviate this decision problem as *ST-Sum*. We have the following theorem:

Theorem 6. *ST-Sum problem is NP-Complete.*

Proof. It is easy to check ST-Sum is in NP. We prove it's NP-hard by showing a polynomial time reduction from Dual-ST-Length to ST-Sum. Given an instance $\langle G, K\rangle$ of Dual-ST-Length, we construct the corresponding instance $\langle G', K'\rangle$ of ST-SUM as follows (see Fig. 5):

- Let s^* be the interior face of G adjacent to the edge (s, t). Insert a path P with $(n-1) = (|V|-1)$ vertices into the face s^*, connecting s and t. Denote this graph by G' (which obviously is a plane graph). We still use (s, t) as the specified exterior edge for G'.
- $K' = K + 1 + n$.

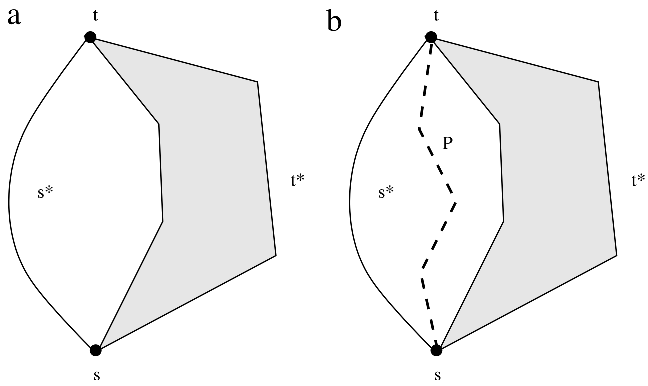

Fig. 5. (a) Graph G; (b) Graph G'.

The construction of $\langle G', K'\rangle$ can easily be done in linear time. We show $\langle G, K\rangle$ is a yes instance of Dual-ST-Length if and only $\langle G', K'\rangle$ is an yes instance of ST-Sum.

Suppose G^* has an st-orientation $\mathcal{O}^*$, with source s^* and sink t^*, such that length$(\mathcal{O}^*) \le K$. Let $\mathcal{O}$ be the corresponding st-orientation for G. From $\mathcal{O}$, we obtain an st-orientation $\mathcal{O}'$ for G' by directing the edges in the path P from s to t. Clearly, length$(\mathcal{O}') = n$ (which is the length of P). The maximum length of any st-path in its dual $\mathcal{O}'^*$ is clearly $K + 1$ because of the split of the face s^*. Hence, length$(O') +$ length$(O'^*) \le n + K + 1 = K'$.

Conversely, suppose that G' has an st-orientation $\mathcal{O}'$ such that the sum of the maximum length of an st-path Q in $G'_{\mathcal{O}'}$ and the maximum length of an st-path Q^* in $G'^*_{\mathcal{O}'^*}$ is $\le K' = K + 1 + n$. Since s is the only source and t is the only sink, Q must start at s and end at t. Hence, the maximum length of an st-path in $\mathcal{O}'$ is exactly n. Therefore, the maximum length of an st-path Q^* in $G'^*_{\mathcal{O}'^*}$ is $\le K' - n = K + 1$. After deleting the edges in P from $\mathcal{O}'$, we obtain an st-orientation $\mathcal{O}$ for G. It is easy to see that, the maximum length of an st-path in $G^*_{\mathcal{O}^*}$ is exactly one less than the maximum length of an st-path in $G'^*_{\mathcal{O}'^*}$ due to the split of face s^* in G. Therefore, G^* has an st-orientation $\mathcal{O}^*$ such that the length of its maximum st-path is $\le K + 1 - 1 = K$. $\square$

5. Conclusion

In this paper, we present a VR construction for 4-connected plane graphs, which simultaneously bounds height $\le \frac{3n}{4} + 2\lceil\sqrt{n}\rceil + 4$ and width $\le \lceil 3n/2\rceil$. This is the first VR construction for 4-connected triangulations that simultaneously bounds the height by $c_h n$ and the width by $c_w n$ where $c_h < 1$ and $c_w < 2$. It would be interesting to find such a VR for broader classes of plane graphs.

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